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Quasistationary quaternionic Hamiltonians and complex stochastic maps

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Abstract

We show that the complex projections of time-dependent η-quasianti-Hermitian quaternionic Hamiltonian dynamics are complex stochastic dynamics in the space of complex quasi-Hermitian density matrices if and only if a quasistationarity condition is fulfilled, i.e. if and only if η is an Hermitian positive time-independent complex operator. An example is also discussed.

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1. Introduction

Studies on non-Hermitian $\mathcal{PT}$-symmetric (or, better, pseudo-Hermitian) Hamiltonians [1] have proven that it is possible to formulate a consistent quantum theory based on such non-Hermitian Hamiltonians [2] at least whenever diagonalizable time-independent Hamiltonians having a real spectrum are taken into account. It was further shown that if the above hypotheses hold, complex quasi-Hermitian systems can be described as open systems, and a master equation was derived [3], proving that the evolution of such systems obeys a one-parameter semigroup law.

Moreover, the theory of open quantum systems can be obtained, in many relevant physical situations, as the complex projection of quaternionic closed quantum systems. (More generally, various hints about the connection between complex and quaternionic maps were recently obtained [4–8].) In particular, it was shown that the complex projection of η-quasianti-Hermitian quaternionic time-independent Hamiltonian dynamics are ruled by one-parameter semigroups of maps in the space of complex quasi-Hermitian density matrices if and only if η is an Hermitian positive *complex* operator [9, 10].

In this paper, we will go more inside into this subject, by considering time-dependent Hamiltonians (we limit ourselves to consider here only the finite-dimensional case). Such a problem was recently investigated [11] in the complex case, and a necessary and sufficient condition was derived for the unitarity of time evolution. We intend here to exploit an analogous method in the quaternionic case, on one hand; on the other hand, we will show that complex stochastic maps (in the sense by Sudarshan *et al* [12]) can be obtained by the complex projection of time-dependent η-quasistationary quaternionic Hamiltonians.

This paper is organized as follows. In section 2 we first recall some previous results on quaternionic pseudo-Hermitian density matrices and then we derive a necessary and sufficient condition for the η-unitarity of the time-evolution associated with an η-pseudoanti-Hermitian quaternionic Hamiltonian. In section 3 we restrict ourselves to consider positive operators η only, and define η-quasistationary quaternionic Hamiltonians. The dynamics ruled by such Hamiltonians are then investigated and explicitly written down, together with their complex projections which constitute stochastic maps in the space of the η-quasi-Hermitian complex density matrices. Such results are illustrated by an example in section 4. Finally, section 5 contains some concluding remarks about the uniqueness of the metric operator and the restrictions imposed on the eigenvectors of the Hamiltonian H in the quasistationary case. Moreover, a hint is given with regard to the construction of the full class of η-quasistationary quaternionic dynamics (hence, of the stochastic maps) associated with a generic time-dependent anti-Hermitian quaternionic operator.

2. Pseudoanti-Hermitian quaternionic Hamiltonian dynamics

In this section, we will introduce the notion of the quaternionic pseudo-Hermitian density matrix and a corresponding Liouville–von Neumann-type equation will be derived.

Denoting by $O^{\ddagger}$ the adjoint of an operator O with respect to the pseudo-inner product

$$(\cdot,\cdot)_\eta = (\cdot, \eta\cdot) \tag{1}$$

(where $(\cdot,\cdot)$ represents the standard quaternionic inner product in the space $\mathbb{Q}^n$), we have

$$O^{\ddagger} = \eta^{-1} O^{\dagger} \eta, \tag{2}$$

so that for any η-pseudo-Hermitian operator, i.e. satisfying the relation

$$\eta O \eta^{-1} = O^{\dagger}, \tag{3}$$

one has $O = O^{\ddagger}$.

If O is η-pseudo-Hermitian, equation (3) immediately implies that ηO is Hermitian, so that the expectation value of O in the state $|\psi\rangle$ with respect to the pseudo-inner product (1) can be obtained:

$$\langle\psi|\,\eta O\,|\psi\rangle = \mathrm{Re}\,\mathrm{Tr}(|\psi\rangle\langle\psi|\eta O) = \mathrm{Re}\,\mathrm{Tr}(\tilde{\rho} O), \tag{4}$$

where $\tilde{\rho} = |\psi\rangle\langle\psi|\eta$.

More generally, if ρ denotes a generic quaternionic density (hence Hermitian and positive) matrix, we can associate with it a *generalized* density matrix $\tilde{\rho}$ (up to a normalization factor N) by means of a one-to-one mapping in the following way:

$$\tilde{\rho} = \frac{1}{N}\rho\eta \tag{5}$$

(where $N = |\mathrm{Re}\,\mathrm{Tr}\,\rho\eta|$ if $\mathrm{Re}\,\mathrm{Tr}\,\rho\eta \neq 0$, otherwise $N = 1$) and obtain $\langle O\rangle_\eta = \mathrm{Re}\,\mathrm{Tr}(\tilde{\rho} O)$.

Note that $\tilde{\rho}$ is η-pseudo-Hermitian:

$$\tilde{\rho}^{\dagger} = \frac{1}{N}\eta\rho = \eta\tilde{\rho}\eta^{-1}.$$

As in the Hermitian case [5–8], equation (4) immediately implies that the expectation value of an η-pseudo-Hermitian operator O on the generalized state $\tilde{\rho}$ depends on the quaternionic parts of O and $\tilde{\rho}$, only if both the operator and the generalized state are represented by genuine quaternionic matrices. Hence, if an η-pseudo-Hermitian operator O is described by a *complex* matrix, its expectation value does not depend on the quaternionic part $\mathrm{j}\tilde{\rho}_\beta$ of the state $\tilde{\rho} = \tilde{\rho}_\alpha + \mathrm{j}\tilde{\rho}_\beta$.

It was shown that whenever the quaternionic Hamiltonian H of a quantum system is η-pseudoanti-Hermitian, i.e.

$$\eta H \eta^{-1} = -H^\dagger, \tag{6}$$

where $\eta = \eta^\dagger$, the pseudo-inner product (1) is invariant under the time translation generated by H provided that η does not depend on t [9]:

$$\langle\psi(t)|\eta|\psi(t)\rangle = \langle\psi(0)|\eta|\psi(0)\rangle. \tag{7}$$

Denoting by $V(t)$ the evolution operator

$$|\psi(t)\rangle = V(t)|\psi(0)\rangle, \tag{8}$$

equation (7) immediately implies

$$V^\dagger \eta V = \eta, \tag{9}$$

i.e. V is η-unitary. Whenever H is time-independent, η-unitarity of V is quite apparent, owing to its explicit form $V(t) = \mathrm{e}^{-Ht}$ ($\hbar = 1$) and invoking η-pseudoanti-Hermiticity of H.

Moreover, from

$$\rho(t) = V\rho(0)V^\dagger \tag{10}$$

by easy calculations, we obtain for a generalized quaternionic density matrix $\tilde{\rho}$

$$\tilde{\rho}(t) = V(t)\tilde{\rho}(0)V(t)^{-1} \tag{11}$$

(where the invertibility of V immediately follows from the invertibility of η in equation (9)). In conclusion, η-unitarity of the time-evolution is a consequence of the η-pseudoanti-Hermiticity of H.

Conversely, let us assume unitarity of the time-evolution with respect to a (possibly time-dependent) η-inner product:

$$\langle\psi(0)|\eta(0)|\phi(0)\rangle = \langle\psi(t)|\eta(t)|\phi(t)\rangle.$$

Then, this condition is equivalent to

$$\eta(0) = V^\dagger(t)\eta(t)V(t).$$

Differentiating both sides of the preceding equation we immediately get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\right)\eta^{-1}(t) = H^\dagger(t) + \eta(t)H(t)\eta(t)^{-1}, \tag{12}$$

where

$$H(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right)V^{-1}(t). \tag{13}$$

Equation (12) shows that $H(t)$ is η-pseudoanti-Hermitian if and only if η is time-independent. In this case, the time-evolution of $\tilde{\rho}(t)$ is described at finite level by equation (11) and at infinitesimal level by the usual Liouville–von Neumann equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}(t) = -[H(t), \tilde{\rho}]. \tag{14}$$

From equation (11), the conservation of the η-pseudo-norm immediately follows

$$\mathrm{Re}\,\mathrm{Tr}\,\tilde{\rho}(t) = \mathrm{Re}\,\mathrm{Tr}\,\tilde{\rho}(0).$$

From equations (11), (9) and the η-pseudo-Hermiticity of $\tilde{\rho}(0)$, we immediately get

$$\eta\tilde{\rho}\,(t)\,\eta^{-1} = \eta V(t)\tilde{\rho}(0)V^{-1}(t)\eta^{-1} = V^{\dagger-1}(t)\eta\tilde{\rho}(0)\eta^{-1}V^{\dagger}(t) = \tilde{\rho}^{\dagger}\,(t)\,,$$

i.e. $\tilde{\rho}\,(t)$ is η-pseudo-Hermitian.

3. Quasistationary quaternionic Hamiltonian dynamics and their complex projections

In this section, we restrict ourselves to consider η-pseudoanti-Hermitian operators associated with a positive metric operator η [14]. While in general the spectrum of an η-pseudoanti-Hermitian operator H can be always assumed to be complex (by suitably rephasing the element of the biorthonormal basis associated with it), in such a special case the spectrum of H is pure imaginary and the Hamiltonian is called quasianti-Hermitian [9]. Hence quasianti-Hermitian Hamiltonians H play in the quaternionic case the role of the quasi-Hermitian ones in complex quantum mechanics.

Let us then consider the space of quaternionic quasi-Hermitian density matrices, that is the subclass of the η-pseudo-Hermitian density matrices with a positive η. Thus, an (Hermitian) operator T exists such that $\eta = T^2$, and the η-pseudo-Hermitian density matrices are positive definite; indeed, putting $\eta = T^2$ into equation (5), from the positivity of ρ we immediately obtain $T\tilde{\rho}T^{-1} = \frac{1}{N}T\rho T = \frac{1}{N}T\rho T^{\dagger} \geqslant 0$ [13].

Then, the inner product (1) we introduced in the Hilbert space is positive, so that all the usual requirements for a proper quantum measurement theory can be maintained [2, 14–16]. Hence, according to the discussion in section 2, the η-unitarity of the time-evolution implies that a time-dependent quaternionic Hamiltonian operator $H(t)$ defines a consistent unitary quaternionic quantum system if and only if $H(t)$ is η-pseudoanti-Hermitian for a time-independent positive η operator. We will call such a Hamiltonian η-quasistationary [11].

When one considers η-quasistationary quaternionic dynamics, any η-unitary operator $V(t)$ can be decomposed as follows:

$$V(t) = T^{-1}U(t)T, \tag{15}$$

where $U^{\dagger}U = \mathbf{1}$.

In fact, by using equation (9) and imposing unitarity, we immediately get

$$T^{-1}V^{\dagger}TTVT^{-1} = \mathbf{1}.$$

Recalling that η is time-independent, we immediately obtain from equations (15), (13)

$$H(t) = T^{-1}\mathfrak{H}(t)T, \tag{16}$$

where $\mathfrak{H}(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}U(t)\right)U^{\dagger}(t)$ is a quaternionic anti-Hermitian operator.

Let us denote by $M(\mathbb{Q})$ and $M(\mathbb{C})$ the space of $n \times m$ quaternionic and complex matrices, respectively, and let $M = M_{\alpha} + \mathrm{j}M_{\beta} \in M(\mathbb{Q})$. We define the complex projection

$$P : M(\mathbb{Q}) \rightarrow M(\mathbb{C})$$

by the relation

$$P[M] = \tfrac{1}{2}[M - \mathrm{i}M\mathrm{i}] = M_{\alpha}. \tag{17}$$

In order to discuss the complex projection of quaternionic η-quasistationary dynamics, we recall the following properties [10]:

(i) the complex projection $\tilde{\rho}_{\alpha}$ of an η-quasi-Hermitian quaternionic matrix $\tilde{\rho} = \tilde{\rho}_{\alpha} + \mathrm{j}\tilde{\rho}_{\beta}$ is η-quasi-Hermitian if and only if the entries of η are complex;

(ii) the complex projection $\tilde{\rho}_\alpha$ *of an* η*-quasi-Hermitian quaternionic matrix* $\tilde{\rho} = \tilde{\rho}_\alpha + \mathrm{j}\tilde{\rho}_\beta$ *with a complex positive* η*, is positive and* $\mathrm{Re}\,\mathrm{Tr}\,\tilde{\rho}_\alpha = 1$.

We sketch here the proof of property (ii). By property (i) it is $\tilde{\rho}_\alpha = \frac{1}{N}\rho_\alpha\eta$. Since $\eta = T^2$, from the positivity of ρ_α we immediately obtain

$$T\tilde{\rho}_\alpha T^{-1} = \frac{1}{N}T\rho_\alpha T^\dagger \geqslant 0,$$

hence $\tilde{\rho}_\alpha \geqslant 0$. Furthermore, trivially, $\mathrm{Re}\,\mathrm{Tr}\,\tilde{\rho}_\alpha = \mathrm{Re}\,\mathrm{Tr}\,\widetilde{\rho} = 1$.

Now, let us consider a dynamics ruled by an η-quasistationary quaternionic Hamiltonian. Equation (11) shows that this dynamics represent a mapping into the set of η-quasi-Hermitian quaternionic density matrices; moreover, if η is complex and positive, properties (i) and (ii) ensure that the complex projection $\tilde{\rho}_\alpha$ of $\tilde{\rho}$ is a complex η-quasi-Hermitian density matrix for any $\tilde{\rho}$. Hence, we can conclude that the complex projection of a η-quasistationary quaternionic Hamiltonian dynamics (with η complex positive) is a complex stochastic dynamics in the space of η-quasi-Hermitian complex density matrices.

The explicit form of such dynamics can be obtained from equations (9)–(11) decomposing V in terms of its complex parts V_α and V_β: $V = V_\alpha + \mathrm{j}V_\beta$. Indeed by equation (10) one has

$$\rho_\alpha(0) \to \rho_\alpha(t) = V_\alpha\rho_\alpha(0)V_\alpha^\dagger + V_\beta^*\rho_\alpha^*(0)V_\beta^T + V_\alpha\rho_\beta^*(0)V_\beta^T - V_\beta^*\rho_\beta(0)V_\alpha^\dagger,$$

which is a complex positive map in the space of Hermitian density matrices ρ_α ($*$ and T in superscript denote the usual complex conjugation and transposition, respectively). It follows that

$$\widetilde{\rho}_\alpha(t) = \frac{1}{N}\left(V_\alpha\rho_\alpha(0)V_\alpha^\dagger + V_\beta^*\rho_\alpha^*(0)V_\beta^T + V_\alpha\rho_\beta^*(0)V_\beta^T - V_\beta^*\rho_\beta(0)V_\alpha^\dagger\right)\eta. \quad (18)$$

Equation (18) can be rewritten in terms of $\widetilde{\rho}_\alpha(0)$ and $\widetilde{\rho}_\beta(0)$ putting $V^{-1} = W_\alpha + \mathrm{j}W_\beta$ and using the relations $V_\alpha^\dagger\eta = \eta W_\alpha$, $-V_\beta^T\eta = \eta^* W_\beta$ (see equation (9))

$$\widetilde{\rho}_\alpha(t) = V_\alpha\widetilde{\rho}_\alpha(0)W_\alpha - V_\beta^*\widetilde{\rho}_\alpha^*(0)W_\beta - V_\alpha\widetilde{\rho}_\beta^*(0)W_\beta - V_\beta^*\widetilde{\rho}_\beta(0)W_\alpha.$$

At an infinitesimal level the previous equation get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\rho}_\alpha = -[H_\alpha, \widetilde{\rho}_\alpha] + H_\beta^*\widetilde{\rho}_\beta - \widetilde{\rho}_\beta^* H_\beta, \quad (19)$$

where the symplectic decomposition of H has been used.

It is worthwhile to stress that, unlike what happens whenever H is time-independent (in such case a one-parameter semigroup dynamics can always be associated with H [10]), in the general case we considered here the evolution operator does not obey a semigroup law, as the example in the following section will show explicitly. Other interesting features of quasistationarity condition will be examined in section 5.

4. An example

Let us first observe that the most general two-dimensional complex positive η operator is given by

$$\eta = T^2 = \begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} = \begin{pmatrix} x^2 + |z|^2 & (x+y)z \\ (x+y)z^* & y^2 + |z|^2 \end{pmatrix}, \quad (20)$$

where $x, y \in \mathbb{R}$, $z \in \mathbb{C}$ and $xy \neq |z|^2$ and

$$T^{-1} = \frac{1}{xy - |z|^2}\begin{pmatrix} y & -z \\ -z^* & x \end{pmatrix}.$$

The complex dynamical map we will study is obtained as the complex projection of a deformation of the quaternionic unitary map,

$$U(t) = \begin{pmatrix} \sqrt{1-(\sin 2t)^4} + \mathrm{j}\,\mathrm{e}^{-\mathrm{i}\theta}(\sin 2t)^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \theta \in \mathbb{R}, \tag{21}$$

to which corresponds the anti-Hermitian time-dependent Hamiltonian

$$\mathfrak{H}(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}U(t)\right)U^{\dagger}(t) = \begin{pmatrix} \mathrm{j}\frac{-4\,\mathrm{e}^{-\mathrm{i}\theta}\sin 2t\cos 2t}{|\cos 2t|\sqrt{1+(\sin 2t)^2}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

We extensively studied such Hamiltonian [8], which generalizes to the quaternionic case a complex stochastic dynamics, arising in some decoherence modeling schemes [12].

From the previous two equations and equations (16), (15), we get

$$H(t) = T^{-1}\mathfrak{H}(t)T = \mathrm{j}\frac{-4\,\mathrm{e}^{-\mathrm{i}\theta}\sin 2t\cos 2t}{(xy-|z|^2)|\cos 2t|\sqrt{1+(\sin 2t)^2}}\begin{pmatrix} yx & yz \\ -zx & -z^2 \end{pmatrix} \tag{23}$$

and

$$V(t) = T^{-1}U(t)T = \frac{1}{xy-|z|^2}\begin{pmatrix} yxq-|z|^2 & y(q-1)z \\ -z^*(q-1)x & -z^*qz+xy \end{pmatrix}, \tag{24}$$

where $q = \sqrt{1-(\sin 2t)^4} + \mathrm{j}\,\mathrm{e}^{-\mathrm{i}\theta}(\sin 2t)^2$.

Let the initial 'pure' state be

$$\tilde\rho(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2(xy-|z|^2)}\mathrm{j}\begin{pmatrix} -(x+y)z^*\,\mathrm{e}^{-\mathrm{i}\theta} & -(|z|^2+y^2)\,\mathrm{e}^{-\mathrm{i}\theta} \\ (|z|^2+x^2)\,\mathrm{e}^{-\mathrm{i}\theta} & (x+y)z\,\mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}, \tag{25}$$

according to equation (11) the final state reads

$$\begin{aligned}\tilde\rho(t) = {} & \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(\sin 2t)^2}{2(xy-|z|^2)}\begin{pmatrix} yz^*-xz & y^2-z^2 \\ (x^2-z^{*2}) & -yz^*+xz \end{pmatrix} \\ & + \mathrm{j}\frac{\mathrm{e}^{-\mathrm{i}\theta}\sqrt{1-(\sin 2t)^4}}{2(xy-|z|^2)}\begin{pmatrix} -z^*(x+y) & -|z|^2-y^2 \\ |z|^2+x^2 & z(x+y) \end{pmatrix}. \end{aligned} \tag{26}$$

The complex projection stochastic dynamics is given by

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(\sin 2t)^2}{2(xy-|z|^2)}\begin{pmatrix} yz^*-xz & y^2-z^2 \\ (x^2-z^{*2}) & -yz^*+xz \end{pmatrix}.$$

Note that the semigroup composition law does not hold. In fact, by a direct computation it is easy to verify that

$$P[V(t)\tilde\rho(0)V^{-1}(t)] \neq P[V(t-t')V(t')\tilde\rho(0)V^{-1}(t')V^{-1}(t-t')],$$

indeed,

$$P[(V(t)\tilde\rho(0)V^{-1}(t))_{21}] = (\sin 2t)^2$$

(where, as usual, M_{ij} denotes the ith row, jth column entry of a matrix M), while

$$\begin{aligned} & P[(V(t-t')V(t')\tilde\rho(0)V^{-1}(t')V^{-1}(t-t'))_{21}] \\ & \quad = (\cos 2(t-t'))^2(\sin 2t')^2 - [(1-(\cos 2(t-t'))^4)(1-(\sin 2t')^4)]^{\frac{1}{2}}. \end{aligned}$$

5. Concluding remarks

Let a Hermitian nonsingular quaternionic operator η be given. Then, the more general η-pseudoanti-Hermitian quaternionic Hamiltonian H can be written in the following way [10]:

$$H = F\eta, \tag{27}$$

where $F^{\dagger} = -F$.

This peculiarity can be useful to obtain the full class of η-quasistationary quaternionic dynamics. In fact, for any time-dependent anti-Hermitian quaternionic operator $F(t)$, we can construct a corresponding anti-Hermitian operator $\mathfrak{H}(t) = \eta^{\frac{1}{2}} F(t) \eta^{\frac{1}{2}}$, and from equation (16) we can state that

$$H(t) = \eta^{-\frac{1}{2}} \mathfrak{H}(t) \eta^{\frac{1}{2}} \tag{28}$$

is a η-quasistationary Hamiltonian, and construct the stochastic (complex) map associated with it, by the methods used in the example above. Conversely, let a η-quasistationary quaternionic dynamics be given, then, a time-dependent anti-Hermitian operator $F(t)$ can be associated with it.

We conclude this paper by analyzing some interesting features of quasistationarity condition. Such features were already exploited in the complex case [11]; we will show now that the same properties hold in the quaternionic case, generalizing the Mostafazadeh approach.

First of all, by a known theorem, when a set of η-quasi-Hermitian (complex) operators is irreducible, then η is uniquely determined (up to a multiplicative constant) [15], and an analogous result was already proven to hold also in quaternionic quantum mechanics [14]. Now, all the elements of the infinite set $\mathfrak{S} = (H_0, H_1, H_2, \ldots)$, where $H_n := \frac{d^n}{dt^n} H(t)|_{t=0}$ are η-quasianti-Hermitian, and assuming that $\mathfrak{S}$ includes an irreducible subset (which is generally true) we deduce unicity of the metric operator η.

In the second place, quasistationarity property imposes severe restrictions on the eigenvectors of H.

In fact, expressing η in terms of the biorthonormal basis $\{\psi_n, \phi_n\}$ associated with H, we have [11, 14]

$$\eta = \sum_n |\phi_n\rangle\langle\phi_n|.$$

Differentiating the above equation, and taking into account biorthonormality, we easily obtain

$$A_{mk}(t) = -A^{Q}_{km}(t),$$

where Q in superscript denotes quaternionic conjugation and

$$A_{mk}(t) := -\langle\phi_m|\frac{d}{dt}|\psi_k\rangle.$$

We conclude that the matrix A is anti-Hermitian, and its diagonal elements are pure imaginary, so that all the features of the complex case hold unchanged in the quaternionic one.

In particular, recalling the Schrödinger equation in quaternionic quantum mechanics ($\hbar = 1$) [18]

$$\frac{d}{dt}|\psi_n\rangle = -H|\psi_n\rangle = -|\psi_n\rangle \mathrm{i}E_n,$$

we have

$$A_{mk}(t) = \mathrm{i}E_k\delta_{mk}.$$

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References

[1] 2004 *Proc. Ist Int. Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics* (Prague, Czech Republic) *Czech. J. Phys.* **54** no 1
2004 *Proc. 2nd Int. Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics* (Prague, Czech Republic) *Czech. J. Phys.* **54** no 10
2005 *Proc. 3rd Int. Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics* (Istanbul, Turkey) *Czech. J. Phys.* **55**
2006 *Proc. IVth Int. Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics* (Stellenbosch, South Africa) *J. Phys. A: Math. Gen.* **39**
2006 *Proc. Vth Int. Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics* (*Czech. J. Phys.* **56**)
[2] Mostafazadeh A 2003 *J. Phys. A: Math. Gen.* **36** 7081
[3] Scolarici G and Solombrino L 2006 *Czech. J. Phys.* **56** 935
[4] Kossakowski A 2000 *Rep. Math. Phys.* **46** 393
[5] Scolarici G and Solombrino L 2006 Complex entanglement and quaternionic separability *The Foundations of Quantum Mechanics: Historical Analysis and Open Questions-Cesena 2004* ed C Garola, A Rossi and S Sozzo (Singapore: World Scientific)
[6] Asorey M and Scolarici G 2006 *J. Phys. A: Math. Gen.* **39** 9727
[7] Asorey M, Scolarici G and Solombrino L 2007 *Theor. Math. Phys.* **151** 733
[8] Asorey M, Scolarici G and Solombrino L 2007 *Phys. Rev.* A **76** 12111
[9] Scolarici G 2002 *J. Phys. A: Math. Gen.* **35** 7493
[10] Scolarici G 2007 *SIGMA* **3** 088
[11] Mostafazadeh A 2007 *Phys. Lett.* B **650** 208
[12] Sudarshan E C G, Mathews P M and Rau J 1961 *Phys. Rev.* **121** 920
[13] Zhang F 1997 *Linear Algebra Appl.* **251** 21
[14] Blasi A, Scolarici G and Solombrino L 2005 *J. Math. Phys.* **46** 42104
[15] Sholtz F G, Geyer H B and Hahne F J W 1992 *Ann. Phys.* **213** 74
[16] Blasi A, Scolarici G and Solombrino L 2004 *J. Phys. A: Math. Gen.* **37** 4335
[17] Rodriguez C A, Shaji A and Sudarshan E C G 2005 Dynamics of two qubits: decoherence and entanglement optimization protocol *Preprint* quant-ph/0504051
[18] Adler S L 1995 *Quaternionic Quantum Mechanics and Quantum Fields* (New York: Oxford University Press)